

SPECIAL SYMPLECTIC SIX-MANIFOLDS

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ABSTRACT. We classify nilmanifolds with an invariant symplectic half-flat structure. We solve the half-flat evolution equations in one example, writing down the resulting Ricci-flat metric. We study the geometry of the orbit space of 6-manifolds with an $SU(3)$ -structure preserved by a $U(1)$ action, giving characterizations in the symplectic half-flat and integrable case.

1. INTRODUCTION

A half-flat manifold is a six-dimensional manifold endowed with an $SU(3)$ -structure whose intrinsic torsion is symmetric. An $SU(3)$ -structure defines a non-degenerate two-form ω , an almost-complex structure J , and a complex volume form Ψ ; the half-flat condition is equivalent to requiring $\omega \wedge \omega$ and the real part of Ψ to be closed [5].

Hypersurfaces in seven-dimensional manifolds with holonomy G_2 have a natural half-flat structure, given by the restriction of the holonomy group representation; the intrinsic torsion can then be identified with the second fundamental form. The converse is not obvious. In [14], Hitchin proved that, starting with a half-flat manifold (M, ω, Ψ) , if certain evolution equations admit a solution coinciding with (ω, Ψ) at time zero, then (M, ω, Ψ) can be embedded isometrically as a hypersurface in a manifold with holonomy contained in G_2 .

Given a six-manifold M with an $SU(3)$ -structure, one can consider the product G_2 -structure on $M \times S^1$. Half-flat manifolds satisfying special conditions on this product G_2 -structure have been studied: in [6] and [4], the six-dimensional nilmanifolds carrying invariant $SU(3)$ -structures of these types have been classified. More generally, the problem of classifying nilmanifolds admitting invariant half-flat structures is open.

In this paper we focus on the symplectic case; that is to say, we take into consideration half-flat structures for which ω is closed. In this context, one can introduce special Lagrangian submanifolds, namely the three-dimensional submanifolds $\iota: L \rightarrow M$ such that both $\iota^*\omega$ and $\iota^*\Im \Psi$ vanish. Like in the integrable case, special Lagrangian submanifolds are exactly the submanifolds calibrated by $\Re \Psi$ [12]. Symplectic half-flat manifolds can also be viewed as symplectic manifolds (M, ω) endowed with two extra objects, namely an ω -calibrated almost-complex structure and a $(3, 0)$ -form with closed real part which is parallel with respect to the Chern connection [8], [9].

Our main result is the classification of nilmanifolds carrying an invariant symplectic half-flat structure; the special case $b_1 \geq 4$ was carried out in [2].

The contents of this paper are organized as follows. Section 2 consists of the classification. Since six-dimensional nilmanifolds can be realized as circle bundles

over nilmanifolds of dimension five, this classification problem can be reduced to a problem in five dimensions: indeed, the symplectic half-flat structure induces an $SU(2)$ -structure on the base of the circle bundle, satisfying certain conditions involving the curvature (see Lemma 2.2). In Lemma 2.3 we classify the five-dimensional nilmanifolds with this type of induced structure. We then use this lemma to show that every six-dimensional nilmanifold admitting a symplectic half-flat structure is modelled on one of a list of three Lie algebras (Theorem 2.4). The corresponding nilmanifolds are the torus, a torus bundle over the four-dimensional torus (see [9] and [11]) and a torus bundle over a three-dimensional torus (see [2] and [11]). In the last case, the fibres are actually special Lagrangian submanifolds.

In Section 3 we fix a symplectic half-flat structure on the irreducible nilmanifold appearing in Theorem 2.4, and solve the evolution equations. We write down the resulting metric, computing the curvature and proving that the holonomy coincides with G_2 . We also observe that if one starts with the reducible example, where the half-flat manifold is the product of a five-manifold with a circle, the resulting seven-dimensional manifold is also reducible.

In Section 4 we generalize this situation and the construction of Lemma 2.2 to the non-invariant case. More precisely, we consider circle bundles with a $U(1)$ -invariant $SU(3)$ -structure, and we define an induced $SU(2)$ -structure on the base; the manifolds are assumed to be compact. The metric underlying the $SU(2)$ -structure is not the same as the quotient metric, but it is obtained from it by rescaling along certain directions, as in [1].

The compactness assumption is dropped in Section 5, where we compute the intrinsic torsion of the $SU(2)$ -structure in terms of the intrinsic torsion of the $SU(3)$ -structure and curvature of the bundle. As far as we know, this is the first detailed application of intrinsic torsion for $SU(2)$ -structures on five-manifolds. We then characterize the $U(1)$ -invariant symplectic half-flat manifolds in terms of the quotient structure (Proposition 5.2). In the case that the $U(1)$ -invariant $SU(3)$ -structure is integrable, we obtain a stronger result: indeed, the intrinsic torsion of the quotient structure and the curvature are determined by the length of the fibres (Theorem 5.3).

2. INVARIANT STRUCTURES ON NILMANIFOLDS

In this section we introduce half-flat structures on 6-manifolds and classify invariant symplectic half-flat structures on 6-dimensional nilmanifolds.

Let M be a 6-dimensional manifold. An $SU(3)$ -structure on M is a pair (ω, Ψ) , where ω is a non-degenerate two-form and Ψ is a decomposable complex three-form, such that the following compatibility conditions hold:

$$(1) \quad \begin{cases} \Psi \wedge \omega = 0 \\ \Psi \wedge \bar{\Psi} = \frac{4}{3}i\omega^3 \end{cases}$$

Indeed, a decomposable complex three-form $\Psi = \theta^1 \wedge \theta^2 \wedge \theta^3$ determines an almost-complex structure J for which θ^1, θ^2 and θ^3 span the space of forms of type $(1, 0)$. The second compatibility condition implies in particular that J is ω -tamed; the first condition asserts that ω is of type $(1, 1)$, so J is actually calibrated by ω .

At each point x of M , the three-form Ψ_x has stabilizer conjugate to $SL(3, \mathbb{C})$ for the natural action of $GL^+(T_x M)$; on the other hand, the stabilizer of ω_x is conjugate

to the symplectic group $\mathrm{Sp}(3, \mathbb{R})$ for the natural action of $\mathrm{GL}(T_x M)$. The compatibility conditions ensure that the intersection of the two stabilizers is conjugate to $\mathrm{SU}(3)$, so that an $\mathrm{SU}(3)$ -structure is defined.

We shall denote by ψ^+ and ψ^- the real and imaginary part of Ψ , respectively. It was shown by Hitchin [13] that having fixed the orientation, the real three-form ψ^+ is sufficient to determine the almost-complex structure J , and therefore ψ^- . Thus, an $\mathrm{SU}(3)$ -structure (ω, Ψ) is really determined by the pair (ω, ψ^+) .

We now introduce a special class of $\mathrm{SU}(3)$ -structures on 6-manifolds, related to 7-dimensional Riemannian manifolds with holonomy contained in G_2 [5]:

Definition 2.1. An $\mathrm{SU}(3)$ -structure (ω, ψ^+) on a 6-manifold is *half-flat* if $\omega \wedge \omega$ and ψ^+ are closed.

The 2-form ω appearing in the characterization of $\mathrm{SU}(3)$ -structures is required to be non-degenerate; if it is also closed, it defines a symplectic structure. In this case, we say that the $\mathrm{SU}(3)$ -structure is symplectic.

Consider a nilmanifold M , i.e. a compact manifold of the form $\Gamma \backslash G$, where G is a 6-dimensional nilpotent group, and Γ a discrete subgroup of G . Recall that in six dimensions, every nilpotent Lie algebra \mathfrak{g} gives rise to such a nilmanifold.

We say that a structure on the nilmanifold M is *invariant* if it pulls back to a left-invariant structure on G . Invariant structures can be viewed as structures on the Lie algebra \mathfrak{g} ; using the above characterization of $\mathrm{SU}(3)$ -structures in terms of differential forms, we shall mainly work with the dual \mathfrak{g}^* .

We start by reducing the problem to a problem in five dimensions; the idea is to realize M as a circle bundle over a 5-dimensional manifold, in such a way that the $\mathrm{SU}(3)$ -structure on M is invariant under the circle action. The geometry of this construction will be studied in Section 4; here, we shall use the following algebraic result:

Lemma 2.2. *Let (ω, ψ^+) be a symplectic half-flat structure on a nilpotent Lie algebra \mathfrak{g} ; we have an orthogonal decomposition*

$$\mathfrak{g}^* = \langle \eta \rangle \oplus V^5 ,$$

where η is a unit form, and

$$d(\mathfrak{g}^*) \subseteq \Lambda^2 V^5 .$$

Define forms α , ω_1 , ω_2 and ω_3 on $\ker \eta$ by

$$(2) \quad \begin{cases} \omega = \omega_3 + \eta \wedge \alpha \\ \Psi = (\omega_1 + i\omega_2) \wedge (\eta + i\alpha) \end{cases}$$

Setting $\phi = d\eta$, the following hold:

$$(3) \quad \begin{cases} d\alpha = 0 , & d\omega_1 = 0 , \\ d\omega_3 = -\phi \wedge \alpha , & d(\omega_2 \wedge \alpha) = \omega_1 \wedge \phi . \end{cases}$$

Proof. By Engel's theorem, some non-zero ξ in \mathfrak{g} satisfies

$$\mathrm{ad}(\xi) = 0 .$$

Choosing for η a suitable multiple of ξ^\flat , and setting $V^5 = \eta^\perp$, the first part of the Lemma is satisfied.

By definition,

$$0 = d\omega = d\omega_3 + \phi \wedge \alpha - \eta \wedge d\alpha ;$$

isolating the component in $\eta \wedge \Lambda^2 V^5$, we deduce that α is closed and $d\omega_3$ satisfies the required equation.

Similarly, the rest of (3) follow from:

$$0 = d\psi^+ = d\omega_1 \wedge \eta + \omega_1 \wedge \phi - d(\omega_2 \wedge \alpha) . \quad \square$$

Remark. The forms (α, ω_i) introduced in Lemma 2.2 define an $SU(2)$ -structure on \mathfrak{g} . More generally, we recall that differential forms $(\alpha, \omega_1, \omega_2, \omega_3)$ on a 5-manifold define an $SU(2)$ -structure if and only if at each point (and hence locally) there exists a coframe e^1, \dots, e^5 such that

$$(4) \quad \begin{cases} \alpha = e^5 & \omega_1 = e^{12} + e^{34} \\ \omega_2 = e^{13} + e^{42} & \omega_3 = e^{14} + e^{23} \end{cases}$$

Here and in the sequel, e^{12} is short for $e^1 \wedge e^2$, and so on. If one fixes an orientation, the condition above is equivalent to the existence of a triplet $(\omega_1, \psi_2, \psi_3)$ with

$$\omega_1 = e^{12} + e^{34} , \quad \psi_2 = e^{135} + e^{425} , \quad \psi_3 = e^{145} + e^{235} .$$

By construction, V^5 is itself the dual of a nilpotent Lie algebra; we shall proceed by listing the 5-dimensional Lie algebras that arise this way. To describe Lie algebras, we shall use symbolic expressions such as

$$\mathfrak{g} = (0, 0, 0, 0, 0, 12) ,$$

meaning that \mathfrak{g}^* has a basis η^1, \dots, η^6 such that $d\eta^6 = \eta^1 \wedge \eta^2$ and η^i is closed for $i = 1, \dots, 5$.

Lemma 2.3. *In the hypotheses of Lemma 2.2, V^5 is one of*

$$(0, 0, 0, 0, 0) , \quad (0, 0, 0, 0, 12) , \quad (0, 0, 0, 12, 13) .$$

Proof. By the same argument we used in the proof of Lemma 2.2, we can construct a filtration

$$V^0 \subset \dots \subset V^5 , \quad \dim V^i = i , \quad dV^{i+1} \subset \Lambda^2 V^i .$$

Moreover, by Lemma 2.2, we can assume that α lies in $V^1 \subset V^4$; therefore, using the fact that the 4-dimensional representation of $SU(2)$ is transitive, we can fix a basis e^1, \dots, e^5 of V^5 satisfying (4), with e^4 in $(V^4)^\perp$.

We have to show that the first Betti number satisfies $b_1 \geq 3$; by the classification of 5-dimensional nilpotent Lie algebras, it will then suffice to show that for some choice of the V^i as above, de^4 lies in $\Lambda^2 V^3$ (in particular, this implies that the step of V^5 is less than two).

By Lemma 2.2,

$$0 = de^{12} + de^{34} \equiv de^3 \wedge e^4 \pmod{\Lambda^2 V^4} ,$$

implying that e^3 is closed. Thus, we can assume

$$V^2 = \langle e^3, e^5 \rangle ; \quad V^4 = \langle e^1, e^2, e^3, e^5 \rangle .$$

Define a real constant h , a 2-form $\gamma \in \Lambda^2 \langle e^1, e^2, e^3 \rangle$ and 1-forms ϕ_4, ϕ_5 in $\langle e^1, e^2, e^3 \rangle$, such that

$$\phi = \phi_5 \wedge e^5 + \phi_4 \wedge e^4 + he^{45} + \gamma .$$

Since ϕ is closed,

$$(5) \quad d\phi_5 \wedge e^5 - \phi_4 \wedge de^4 + hde^4 \wedge e^5 + d\gamma = -d\phi_4 \wedge e^4 ;$$

the left-hand side lies in $\Lambda^3 V^4$, i.e. it has no component containing e^4 , so both sides are zero and $d\phi_4 = 0$.

By Lemma 2.2 $d\omega_3 = -\phi \wedge \alpha$, giving

$$(6) \quad de^{14} + de^2 \wedge e^3 = -\phi_4 \wedge e^{45} - \gamma \wedge e^5 ;$$

comparing the components in $e^4 \wedge \Lambda^2 V^4$, we obtain

$$de^1 = \phi_4 \wedge e^5 .$$

Again by Lemma 2.2, $d\psi_2 = \omega_1 \wedge \phi$; on the other hand, $d\psi_2 = d\omega_2 \wedge \alpha$, so we can drop the component of ϕ not containing e^5 and write

$$(7) \quad (de^{13} + de^{42}) \wedge e^5 = (e^{12} + e^{34}) \wedge (\phi_5 \wedge e^5 + he^{45}) .$$

The components containing e^4 give

$$de^2 \wedge e^5 = e^3 \wedge \phi_5 \wedge e^5 - he^{125} ,$$

and wedging with e^3 ,

$$de^2 \wedge e^{35} = -he^{1235} .$$

Since de^2 is in $\Lambda^2 V^3$, the left hand side is in $\Lambda^4 V^3$, so it is zero. We conclude that $h = 0$ and

$$(8) \quad de^2 \wedge e^5 = e^3 \wedge \phi_5 \wedge e^5 ;$$

now, either $V^3 = \langle e^1, e^3, e^5 \rangle$, or some linear combination $\lambda e^1 + e^2$ lies in V^3 , and consequently $0 = (\lambda de^1 + de^2) \wedge e^5 = de^2 \wedge e^5$. Either way,

$$\phi_5 \in \langle e^1, e^3 \rangle .$$

By Lemma 2.2 ω_1 is closed, giving

$$(9) \quad \phi_4 \wedge e^{52} - e^1 \wedge de^2 = e^3 \wedge de^4 ;$$

to proceed further, we must distinguish three cases.

i) Suppose that ϕ_4 is not a multiple of e^3 ; then

$$V^3 = \langle e^3, e^5, \phi_4 \rangle ,$$

and d is zero on V^3 . Moreover e^1 is not closed, so $V^4 = V^3 \oplus \langle e^1 \rangle$. Since e^2 is in V^4 , we have $de^2 = k e^5 \wedge \phi_4$ for some (possibly zero) constant k . So (9) becomes

$$\phi_4 \wedge e^{52} - k e^{15} \wedge \phi_4 = e^3 \wedge de^4 ,$$

implying that

$$de^4 \wedge e^3 \wedge \phi_4 = 0 = de^4 \wedge e^{35} .$$

Since the space of closed two-forms in $\Lambda^2 V^4$ is

$$\Lambda^2 V^3 \oplus \langle e^{15}, e^1 \wedge \gamma_4 \rangle ,$$

we can conclude that de^4 lies in $\Lambda^2 V^3$; we already know that $b_1 \geq 3$, so there is nothing left to prove in this case.

In general, the component of (6) in $\Lambda^3 V^4$ gives

$$(10) \quad -e^1 \wedge de^4 + de^2 \wedge e^3 = -\gamma \wedge e^5 ;$$

in particular, $de^4 \wedge e^{15} = 0$. Moreover, we can rewrite (5) as

$$(11) \quad -\phi_4 \wedge de^4 + d\gamma = 0 .$$

ii) Suppose now that $\phi_4 = 0$; then e^1 is closed and we can assume that $V^3 = \langle e^1, e^3, e^5 \rangle$. By (9),

$$-e^1 \wedge de^2 = e^3 \wedge de^4 ,$$

so clearly $de^4 \wedge e^{13} = 0$; moreover $de^4 \wedge e^{35} = 0$, since by (8) $de^2 \wedge e^{15}$ is zero. It follows that de^4 lies in $\Lambda^2 V^3$, completing the proof in this case.

iii) The remaining case is the one where $\phi_4 = ae^3$ for some non-zero a . By (11) and (9), this condition implies

$$(12) \quad d\gamma = ae^3 \wedge de^4 = a^2 e^{352} - ae^1 \wedge de^2 .$$

Equations (8) and (9) show that

$$de^2 \wedge e^{15} = 0 = de^2 \wedge e^{13} = de^2 \wedge e^{35} ,$$

so de^2 lies in $\Lambda^2 \langle e^1, e^3, e^5 \rangle$. Hence the space of closed forms in $\Lambda^2 V^4$ is contained in

$$\Lambda^2 \langle e^1, e^3, e^5 \rangle \oplus e^2 \wedge \langle e^3, e^5 \rangle ;$$

wedging the closed two-form $\gamma - ae^{12}$ with e^{35} we deduce

$$\gamma \wedge e^{35} = ae^{1235} .$$

Comparing with (10), we find $e^{13} \wedge de^4 = -ae^{1235}$, which together with (12) gives the contradiction

$$-a^2 e^{1235} = a^2 e^{1352} . \quad \square$$

All three possibilities listed in Lemma 2.3 can occur:

- On $V^5 = (0, 0, 0, 0, 0)$, set

$$\omega_1 = \eta^{12} + \eta^{34} , \quad \psi_2 = \eta^{135} + \eta^{425} , \quad \psi_3 = \eta^{145} + \eta^{235} , \quad \phi = 0 .$$

- On $V^5 = (0, 0, 0, 0, 12)$, set

$$\omega_1 = \eta^{34} + \eta^{15} , \quad \psi_2 = \eta^{312} + \eta^{542} , \quad \psi_3 = \eta^{352} + \eta^{412} , \quad \phi = -\eta^{13} .$$

- On $V^5 = (0, 0, 0, 12, 13)$, set

$$\omega_1 = \eta^{24} + \eta^{35} , \quad \psi_2 = \eta^{123} + \eta^{154} , \quad \psi_3 = \eta^{125} + \eta^{143} , \quad \phi = -2\eta^{23} ; \text{ or}$$

$$\omega_1 = \eta^{24} - \eta^{35} , \quad \psi_2 = -\eta^{123} + \eta^{154} , \quad \psi_3 = \eta^{125} - \eta^{143} , \quad \phi = 0 .$$

It is easy to verify that Equations 3 are satisfied in these cases. The construction can then be inverted: define

$$\mathfrak{g}^* = \langle \eta \rangle \oplus V^5 ,$$

declaring that $d\eta = \phi$; clearly, \mathfrak{g}^* is the dual of a nilpotent Lie algebra \mathfrak{g} . A straightforward calculation shows that the $SU(3)$ -structure on \mathfrak{g} defined by (2) is half-flat and symplectic.

So, there are three non-isomorphic nilpotent Lie algebras that admit a symplectic half-flat structure. It only remains to show that this list is complete.

Theorem 2.4. *The 6-dimensional nilpotent Lie algebras whose corresponding nil-manifold carries an invariant symplectic half-flat structure are*

$$(0, 0, 0, 0, 0, 0) , \quad (0, 0, 0, 0, 12, 13) , \quad (0, 0, 0, 12, 13, 23) .$$

Proof. We retain the notation from the proof of Lemma 2.3. We first show that ϕ_4 is zero; in other words, case *i*) of Lemma 2.3 cannot occur, like case *iii*), which we have already ruled out. Indeed, suppose that ϕ_4 is independent of e^3 and e^5 . Hence $V^5 = (0, 0, 0, 12, 13)$; indeed, de^1 and de^4 must be independent, since otherwise a combination of e^1 and e^4 would lie in $\ker d$, which is orthogonal to e^4 by construction.

Observe that $\langle e^1, e^3, \phi^4 \rangle$ has dimension three, because ϕ_4 is closed but e^1 is not, and we are assuming that ϕ_4 is not a multiple of e^3 . Therefore,

$$\langle e^1, e^3, \phi^4 \rangle = \langle e^1, e^2, e^3 \rangle ,$$

and we can write $\gamma = a e^{13} + b e^1 \wedge \phi_4 + c e^3 \wedge \phi_4$; Equation 10 then yields

$$-e^1 \wedge de^4 + k e^5 \wedge \phi_4 \wedge e^3 = -a e^{135} + b e^{15} \wedge \phi_4 + c e^{35} \wedge \phi_4 ,$$

so in particular

$$de^4 = a e^{35} - b e^5 \wedge \phi_4 ;$$

substituting in (11), it follows that

$$-a \phi_4 \wedge e^{35} + a \phi_4 \wedge e^{53} = 0 ,$$

i.e. $a = 0$; but then $de^4 = b e^5 \wedge \phi_4$ is a multiple of de^1 , which is absurd.

We have proved that ϕ_4 is necessarily zero; now assume that e^2 is closed. Then (9) and (10) give

$$e^3 \wedge de^4 = 0 , \quad e^1 \wedge de^4 = \gamma \wedge e^5 .$$

It follows that $de^4 = \lambda e^{35} + \mu e^{13}$ and $\gamma = \lambda e^{13}$ for some constants λ and μ . The components of (7) not containing e^4 give

$$\mu e^{1325} = -e^{125} \wedge \phi_5 ;$$

Since as a consequence of (8) ϕ_5 is a multiple of e^3 , it follows that $\phi_5 = -\mu e^3$. Summing up,

$$\phi = -\mu e^{35} + \lambda e^{13} ,$$

so that ϕ and de^4 are either linearly independent or both zero. The resulting 6-dimensional Lie algebras are

$$(0, 0, 0, 0, 12, 13) , \quad (0, 0, 0, 0, 0, 0) .$$

If e^2 is not closed, V^5 is $(0, 0, 0, 12, 13)$. As both de^4 and de^2 are in $\Lambda^2 V^3$, (10) implies that γ is a multiple of e^{13} . Therefore ϕ lies in $\Lambda^2 V^3$ as well, forcing \mathfrak{g} to be either $(0, 0, 0, 12, 13, 23)$ or $(0, 0, 0, 0, 12, 13)$. \square

3. ASSOCIATED RICCI-FLAT METRICS

In this section we show that the symplectic half-flat manifolds of Theorem 2.4 can be realized as hypersurfaces in Ricci-flat manifolds; in one example, we compute explicitly the metric and its curvature, proving that the holonomy is G_2 .

Recall that a G_2 -structure on a 7-manifold N is defined by a three-form φ which at each point x lies in the $GL(T_x N)$ orbit of

$$(13) \quad e^{147} + e^{257} + e^{367} + e^{123} - e^{156} - e^{426} - e^{453} ,$$

where e^1, \dots, e^7 is any coframe at x . Since G_2 is contained in $SO(7)$, φ determines a metric and an orientation on N .

The G_2 -structure defined by φ is integrable if and only if φ is closed and co-closed [10]. Then the corresponding Riemannian metric has holonomy contained in G_2

and is therefore Ricci-flat.

Now let N be a manifold with an integrable G_2 -structure and let $\iota: M \rightarrow N$ be a hypersurface. Then there exists a unique $SU(3)$ -structure (ω, ψ^+) on M such that

$$\psi^+ = \iota^* \varphi, \quad \omega^2 = 2\iota^* * \varphi;$$

from the integrability of the G_2 -structure it clearly follows that this induced structure is half-flat. Conversely, it is known that if $(\omega(t), \psi^+(t))$ is a one-parameter family of half-flat structures on M , for t ranging in (a, b) , then

$$(14) \quad \varphi = \omega \wedge dt + \psi^+$$

defines a G_2 -structure on $M \times (a, b)$, which is integrable if and only if $(\omega(t), \psi^+(t))$ satisfies the *half-flat evolution equations*:

$$(15) \quad d\omega = \frac{\partial}{\partial t} \psi^+, \quad d\psi^+ = -\frac{1}{2} \frac{\partial}{\partial t} \omega^2.$$

In this construction, $(\omega(t), \psi^+(t))$ must satisfy the compatibility conditions (1) for all t . However, it turns out that ψ^- is still defined for small deformations of an $SU(3)$ -structure, and if (15) are satisfied, then (1) are preserved in time. Indeed, we have the following [14]:

Theorem 3.1 (Hitchin). *Let M be a compact 6-manifold, and let $(\omega(t), \psi^+(t))$ be a one-parameter family of sections of $\Lambda^2(M) \oplus \Lambda^3(M)$ satisfying the evolution equations (15). If $(\omega(0), \psi^+(0))$ is a half-flat structure, and $\omega(t)^3$ is nowhere zero for t in (a, b) , then $(\omega(t), \psi^+(t))$ defines a half-flat structure for all $t \in (a, b)$. In particular, $M \times (a, b)$ has a Riemannian metric with holonomy contained in G_2 .*

We now solve the evolution equations (15) for the nilmanifold with Lie algebra $(0, 0, 0, 12, 13, 23)$. Our solution is different from the one given in [4], since we choose symplectic initial data. Consider the one-parameter family of $SU(3)$ -structures given by

$$(16) \quad \begin{cases} \omega = \frac{1}{u} \eta^{16} - \frac{1}{u} \eta^{25} - \frac{3u^2 - 1}{u} \eta^{34} \\ \psi^+ = \frac{(3u^2 - 1)^2}{4u^6} \eta^{123} - 2\eta^{154} + 2\eta^{624} + \eta^{653} \end{cases}$$

Clearly, (ω, ψ^+) is half-flat for all values of u , and symplectic for $u = \pm 1$. Setting

$$t = -12 + \frac{1}{2u^3} - \frac{1}{10u^5},$$

Equations 16 give a solution of (15). An orthonormal basis of 1-forms is given by

$$\begin{aligned} E^1 &= \sqrt{\frac{3u^2 - 1}{2u^4}} \eta^1, & E^2 &= \sqrt{\frac{3u^2 - 1}{2u^4}} \eta^2, & E^3 &= \frac{3u^2 - 1}{2u^2} \eta^3, \\ E^4 &= \sqrt{\frac{2u^2}{3u^2 - 1}} \eta^6, & E^5 &= -\sqrt{\frac{2u^2}{3u^2 - 1}} \eta^5, & E^6 &= -2u \eta^4, \end{aligned}$$

where indices and signs have been adjusted for compatibility with (13). Let N be the 7-dimensional Riemannian manifold obtained by the above construction; set $E^7 = dt$, and consider the inclusion $\Lambda^2(N) \subset \text{End}(TN)$, where the two-form E^{ij} is

identified with the skew-symmetric endomorphism mapping E^i to E^j . As a section of $S^2(\Lambda^2(N))$, the curvature is given by

$$\begin{aligned} & -\frac{4u^{10}}{(3u^2-1)^4} \left(3(E^{17}+E^{35})^2 + 3(E^{34}-E^{27})^2 + (E^{14}-E^{25})^2 - (E^{12}+E^{45})^2 \right) + \\ & -\frac{12u^{10}(2u^2-1)}{(3u^2-1)^3} \left((E^{16}+E^{27})^2 + (E^{17}-E^{26})^2 - 2(E^{12}-E^{67})^2 \right) + \\ & +\frac{12u^{10}(u^2-1)}{(3u^2-1)^4} \left((E^{24}+E^{37})^2 + (E^{15}-E^{37})^2 \right) + \\ & -\frac{12u^{10}(u^2-2)}{(3u^2-1)^4} \left((E^{13}+E^{57})^2 + (E^{23}-E^{47})^2 \right) + \\ & -\frac{4u^{10}}{(3u^2-1)^3} \left((E^{23}+E^{56})^2 + (E^{13}+E^{46})^2 - (E^{14}-E^{36})^2 \right). \end{aligned}$$

This shows that the metric is not reducible and the holonomy equals G_2 .

Similar computations for a symplectic half-flat structure on the Lie algebra $\mathfrak{g} = (0, 0, 0, 0, 12, 13)$ were carried out in [7]. In this case though, as one can see by computing the curvature, the resulting 7-manifold is the product of a 6-manifold with holonomy $SU(3)$ and a circle. In fact, the nilmanifold is a trivial circle bundle with connection form $\eta = \eta^4$. The symplectic half-flat structure induces an $SU(2)$ -structure (α, ω_i) on the five-dimensional base by (2); this structure is *hypo* in the sense of [7] (see also Section 4), and can therefore be evolved to give a 6-manifold with holonomy $SU(3)$. The reducible 7-manifold is nothing but the product of this 6-manifold with a circle.

Remark. Whilst by Hitchin's theorem the half-flat conditions are automatically preserved in time, the symplectic condition is not, as shown in the above example. This is a general fact: the evolution flow is transverse to the space of symplectic half-flat structures, except where it vanishes (namely, at points defining integrable structures).

4. $SU(3)$ -STRUCTURES ON CIRCLE BUNDLES

In this section we pursue an idea introduced in Section 2, namely that of reducing a 6-dimensional manifold to a 5-dimensional manifold by means of a quotient, and establishing a relation between the two geometries in terms of G -structures. Here we work in a more general context, without requiring invariance under a transitive action; however, for the construction to make sense we still need invariance along one direction, i.e. a Killing field on the 6-manifold. More precisely, we shall establish a one-to-one correspondence between a class of 6-manifolds with an $SU(3)$ -structure and a regular vector field preserving the structure, and a class of 5-manifolds with an $SU(2)$ -structure plus some additional data; since this correspondence only holds “up to isomorphism”, it will be natural to state it in terms of categories. For the moment, we impose no integrability conditions on the $SU(3)$ -structure.

We define a category \mathcal{K} whose objects are 4-tuples (M, ω, ψ^+, X) , where M is a compact 6-dimensional manifold, (ω, ψ^+) is an $SU(3)$ -structure on M , and X is a regular vector field on M which preserves the $SU(3)$ -structure, i.e.

$$\mathcal{L}_X \omega = 0 = \mathcal{L}_X \psi^+.$$

For brevity, we shall often write M for (M, ω, ψ^+, X) ; so, when M is referred to as an object of \mathcal{K} , it will be understood that ω , ψ^+ and X are also fixed on M . Sometimes we will need to consider two distinct objects M , \tilde{M} , and it will be understood that \tilde{M} stands for $(\tilde{M}, \tilde{\omega}, \tilde{\psi}^+, \tilde{X})$.

A morphism $f \in \text{Hom}(M, \tilde{M})$ is a smooth map $f: M \rightarrow \tilde{M}$ such that

$$f^* \tilde{\omega} = \omega, \quad f^* \tilde{\psi}^+ = \psi^+, \quad X \text{ is } f\text{-related to } \tilde{X}.$$

In particular, morphisms are orientation-preserving isometries, and therefore covering maps.

We are going to relate \mathcal{K} to a category \mathcal{C} , whose objects are 6-tuples

$$(N, \alpha, \omega_1, \omega_2, \omega_3, \phi, t),$$

where N is a compact 5-manifold, (α, ω_i) is an $\text{SU}(2)$ -structure on N , t is a function on N , and ϕ is a closed two-form on N such that

$$\left[\frac{1}{2\pi} \phi \right] \in H^2(N, \mathbb{Z}).$$

Again, we shall write N for an object $(N, \alpha, \omega_1, \omega_2, \omega_3, \phi, t)$ of \mathcal{C} .

A morphism $f \in \text{Hom}(N, \tilde{N})$ is a smooth map $f: N \rightarrow \tilde{N}$ such that

$$f^* \tilde{\alpha} = \alpha, \quad f^* \tilde{\omega}_i = \omega_i, \quad i = 1, 2, 3 \quad f^* \tilde{\phi} = \phi, \quad \tilde{t} \circ f = t.$$

We shall construct a functor $F: \mathcal{K} \rightarrow \mathcal{C}$ which realizes each object M of \mathcal{K} as a circle bundle over $F(M)$; the 2-form ϕ represents the curvature, and the function t the length of the fibres.

Let us first recall that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is *faithful* (resp. *full*) if

$$F: \text{Hom}(A, B) \rightarrow \text{Hom}(F(A), F(B))$$

is one-to-one (resp. onto) for all objects A, B of \mathcal{C} ; it is *representative* if every object in \mathcal{D} is isomorphic to $F(A)$ for some object A of \mathcal{C} . A full, representative, faithful functor is called an *equivalence*.

The functor we consider does not quite establish an equivalence, but we shall show that it induces an equivalence between categories derived from \mathcal{K} and \mathcal{C} . The starting point is the following observation: if M is an object of \mathcal{K} , the maximal integral curves of X are closed subsets of the compact manifold M , and therefore diffeomorphic to circles. More precisely, we have the following [3, p. 27]:

Lemma 4.1. *For every object M of \mathcal{K} , the maximal integral curves of X viewed as maps $\phi_x: \mathbb{R} \rightarrow M$ are periodic, with period not depending on the point x . In particular M is the total space of a circle bundle, and X is a fundamental vector field.*

We can now prove the following:

Proposition 4.2. *There is a representative functor $F: \mathcal{K} \rightarrow \mathcal{C}$.*

Proof. We define the covariant functor $F: \mathcal{K} \rightarrow \mathcal{C}$ as follows: let M be an object of \mathcal{K} . The space of integral lines of X is a compact 5-manifold N , and by Lemma 4.1 M is the total space of a circle bundle over N . Since X is a Killing vector field, the norm of X is constant on the fibres, and so defines a smooth function t on N .

By Lemma 4.1, the maximal integral curves of X have constant period; we can rescale X so that this period is 2π . Let η be the connection 1-form determined

by X , i.e. the dual form to X rescaled so that $\eta(X) = 1$; set $\phi = d\eta$. Then the cohomology class

$$c_1 = \left[\frac{1}{2\pi} \phi \right]$$

is the Chern class of the $U(1)$ -bundle $M \rightarrow N$; as such, it is integral. Define forms (α, ω_i) on M by

$$(17) \quad \begin{cases} \alpha = X \lrcorner \omega, & \omega_3 = tX \lrcorner (\omega \wedge \eta), \\ \omega_1 = X \lrcorner \psi^+, & \omega_2 = X \lrcorner \psi^-. \end{cases}$$

By construction, all the objects appearing on the right-hand sides of (17) are invariant under the action of $U(1)$; therefore, each form $\alpha, \omega_1, \omega_2, \omega_3$ is the pullback of a form on N , which we denote by the same symbol.

Now choose a local orthonormal basis of 1-forms

$$\frac{1}{\sqrt{t}}e^1, \dots, \frac{1}{\sqrt{t}}e^4, \frac{1}{t}e^5, e^6$$

on M , with $\eta = t^{-1}e^6$, such that

$$\omega = \frac{1}{t}(e^{14} + e^{23} + e^{65}), \quad \Psi = \frac{1}{t}(e^1 + ie^4) \wedge (e^2 + ie^3) \wedge (e^6 + \frac{1}{t}ie^5).$$

Then (4) is satisfied, so (α, ω_i) defines an $SU(2)$ -structure on N . In particular, e^1, \dots, e^5 is an orthonormal basis of 1-forms on N : so, we are not using the quotient metric on the 5-manifold, but a deformation of it.

If M and \tilde{M} are objects of \mathcal{K} and $f \in \text{Hom}(M, \tilde{M})$, then f maps integral curves of X to integral curves of \tilde{X} . Therefore, f induces a smooth map $F(f): N \rightarrow \tilde{N}$, where $N = F(M)$, $\tilde{N} = F(\tilde{M})$; we must show that $F(f)$ is a morphism. From the fact that X is f -related to \tilde{X} and f is a local isometry, it follows that $\tilde{t} \circ F(f) = t$. Now consider the diagram of maps

$$\begin{array}{ccc} M & \xrightarrow{f} & \tilde{M} \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ N & \xrightarrow{F(f)} & \tilde{N} \end{array}$$

By the commutativity of the diagram and (17),

$$\pi^*(F(f)^*\tilde{\alpha}) = f^*(\tilde{\pi}^*\tilde{\alpha}) = \alpha;$$

therefore, $F(f)^*\tilde{\alpha} = \alpha$ on N . By the same argument $F(f)^*\tilde{\omega}_i = \omega_i$ and $F(f)^*\tilde{\phi} = \phi$.

Next we show that F is representative; more precisely, that every object N of \mathcal{C} can be written as $F(M)$ for some object M of \mathcal{K} . Indeed, let M be a circle bundle over N with Chern class $[\phi/(2\pi)]$. Let A be the standard generator of the Lie algebra $\mathfrak{u}(1)$, and let $X = A^*$ be the associated fundamental vector field. Choose a connection form η such that $d\eta = \phi$ and define

$$(18) \quad \begin{cases} \omega = t^{-1}\omega_3 + \eta \wedge \alpha \\ \Psi = (\omega_1 + i\omega_2) \wedge (\eta + it^{-2}\alpha) \end{cases}$$

It is clear that this defines an $SU(3)$ -structure preserved by X , which has norm $t\eta(X) = t$, and Equations 17 give back the original $SU(2)$ -structure on N . \square

Remark. One might wonder at the advantage of referring explicitly to categories, rather than just defining F and studying its properties. With the latter approach, a problem arises when one tries to construct an inverse to F : indeed, “the” circle bundle with a given Chern class is not a well-defined manifold, but something only defined up to isomorphism.

Remark. The reduction we have chosen behaves well with respect to evolution theory. Indeed, let N be a 5-manifold with an $SU(2)$ -structure; recall that N is called *hypo* if the forms ω_1 , $\omega_2 \wedge \alpha$, and $\omega_3 \wedge \alpha$ are closed. It is easy to verify that the $SU(3)$ -structure induced on $N \times S^1$ by the above construction (corresponding to taking $\phi = 0$ and $t = 1$) is half-flat if and only if the structure on N is hypo. Moreover, hypo geometry also has evolution equations similar to (15), and it turns out that a one-parameter solution of the hypo evolution equations lifts to a solution of the half-flat evolution equations. An example of this situation is the reducible half-flat nilmanifold mentioned in Section 3.

The functor F fails to be an equivalence in two respects: it is not full, and it is not faithful. We start by addressing the first issue. Let \mathcal{C}' be the subcategory of \mathcal{C} consisting of objects N such that N is simply-connected as a manifold; let \mathcal{K}' be the subcategory of \mathcal{K} of objects M such that $F(M)$ is simply-connected.

Lemma 4.3. *The functor $F: \mathcal{K}' \rightarrow \mathcal{C}'$ is full.*

Proof. Consider two objects M, \tilde{M} in \mathcal{K}' ; let $F(M) = N$, $F(\tilde{M}) = \tilde{N}$, and let $h \in \text{Hom}(N, \tilde{N})$. Fix a point u in M ; every path σ in N based at $\pi(u)$ has a horizontal lift γ with $\gamma(0) = u$. Fix also a point \tilde{u} in \tilde{M} , lying over $\pi(u)$; then σ lifts to $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{u}$. Now define $f(\gamma(t)) = \tilde{\gamma}(t)$; we have to show that the definition does not depend on the path σ .

Indeed, let $I = [0, 1]$, and let $\Sigma: I \times I \rightarrow N$ be a smooth homotopy with fixed endpoints between two paths Σ_0 and Σ_1 starting at $\pi(u)$, where $\Sigma_t = \Sigma(t, \cdot)$. The pullback bundle Σ^*M is trivial; we can therefore choose a section

$$\Gamma: I \times I \rightarrow M,$$

where Γ_0 and Γ_1 are horizontal lifts of Σ_0, Σ_1 respectively, and $\Gamma(\cdot, 0) = u$.

The “vertical distance” from $\Gamma_0(1)$ to $\Gamma_1(1)$ is measured by

$$\int_{\Gamma(\cdot, 1)} \eta = \int_{I \times I} \Gamma^* d\eta = \int_{I \times I} \Sigma^* \phi,$$

where we have used Stokes’ theorem. Quite similarly if $\tilde{\Sigma} = h \circ \Sigma$ and $\tilde{\Gamma}$ is constructed as above, imposing this time $\tilde{\Gamma}(\cdot, 0) = \tilde{u}$, we obtain

$$\int_{\tilde{\Gamma}(\cdot, 1)} \tilde{\eta} = \int_{I \times I} \tilde{\Gamma}^* d\tilde{\eta} = \int_{I \times I} \tilde{\Sigma}^* \tilde{\phi} = \int_{I \times I} \Sigma^* \phi,$$

because $h^* \tilde{\phi} = \phi$.

Hence, we can lift h to an equivariant map $f: M \rightarrow \tilde{M}$ satisfying $f^* \tilde{\eta} = \eta$. Thus,

$$F: \text{Hom}(M, \tilde{M}) \rightarrow \text{Hom}(N, \tilde{N})$$

is onto as required. \square

Remark. The definition of the lift f in the proof of Lemma 4.3 is also a characterization, because morphisms preserve the connection form, and therefore map horizontal paths to horizontal paths. So, in the non-simply-connected case one

cannot expect to be able to produce a lift, i.e. F is not full as a functor from \mathcal{K} to \mathcal{C} .

We now come to faithfulness. Fix an object M in \mathcal{K}' and consider the periodic integral lines ϕ_x ; the map

$$M \ni x \xrightarrow{f_\theta} \phi_x(\theta) \in M$$

is an isomorphism for all constants θ . Clearly, $F(f_\theta)$ is the identity; so,

$$F: \text{Hom}(M, M) \rightarrow \text{Hom}(F(M), F(M))$$

is not one-to-one and F is not faithful.

However, this is the only amount to which F fails to be an equivalence. Indeed, for each M, \tilde{M} in \mathcal{K}' , define an equivalence relation in $\text{Hom}(M, \tilde{M})$ by

$$f_1 \sim f_2 \iff f_1 = f_2 \circ f_\theta, \theta \in \mathbb{R}.$$

We can then consider the *quotient category* \mathcal{K}'' , whose objects are the objects of \mathcal{K}' , and whose morphisms are defined by

$$\text{Hom}_{\mathcal{K}''}(M, \tilde{M}) = \text{Hom}_{\mathcal{K}'}(M, \tilde{M}) / \sim.$$

Proposition 4.4. *\mathcal{K}'' is a category, and the induced functor $F'': \mathcal{K}'' \rightarrow \mathcal{C}'$ is an equivalence.*

Proof. By construction, elements of $\text{Hom}(M, \tilde{M})$ are $U(1)$ -equivariant. Therefore,

$$(f_1 \circ f_{\theta_1}) \circ (f_2 \circ f_{\theta_2}) = f_1 \circ f_2 \circ f_{\theta_1 + \theta_2}.$$

It follows that if $g_1 \sim f_1$ and $g_2 \sim f_2$ then $g_1 \circ g_2 \sim f_1 \circ f_2$. This is sufficient to conclude that \mathcal{K}'' is a category.

Observe that the induced functor F'' is well defined, because clearly $f_1 \sim f_2$ implies $F''(f_1) = F''(f_2)$.

Now, recall from the proof of Proposition 4.2 that the connection form is defined only by the metric and the Killing field; since a morphism $f \in \text{Hom}(M, \tilde{M})$ is an isometry and X is f -related to \tilde{X} , we have $f^*\tilde{\eta} = \eta$. Therefore, f is uniquely determined by its value at a point, or in other words,

$$F'': \text{Hom}_{\mathcal{K}''}(M, M) \rightarrow \text{Hom}(F''(M), F''(M))$$

is one-to-one, as required. \square

5. INTRINSIC TORSION OF THE QUOTIENT STRUCTURE

In this section we drop the assumptions of compactness and global regularity, and study the local behaviour of the construction of Section 4 in terms of intrinsic torsion. In particular, we characterize the intrinsic torsion of the $SU(2)$ -structures obtained by taking the quotient of a symplectic half-flat structure, generalizing Lemma 2.2. Then, in the assumption that the starting $SU(3)$ -structure is integrable, we write down a differential equation that the function t must satisfy, and prove that the intrinsic torsion of the quotient $SU(2)$ -structure depends only on t . More precisely, we give necessary and sufficient conditions on (N, α, ω_i, t) for it to arise, locally, as the quotient of a 6-manifold with an integrable $SU(3)$ -structure. Observe that in the integrable case the 6-manifold cannot be compact, unless it is reducible.

We shall work in a neighbourhood of a point where the Killing field is non-zero; thus, we assume that M is a 6-manifold with an $SU(3)$ -structure preserved by some regular Killing field X . Recall from the proof of Proposition 4.2 that the quotient

N is a 5-manifold on which an $SU(2)$ -structure (α, ω_i) is induced (see (17)), as well as a function t , the norm of X , and a two-form ϕ , which in the case of a circle bundle is the curvature form.

Recall from [5] that the intrinsic torsion of an $SU(3)$ -structure takes values in a 42-dimensional space, and its components can be represented as follows:

$$(19) \quad \begin{array}{|c|c|} \hline W_1^+ & W_1^- \\ \hline W_2^+ & W_2^- \\ \hline W_3 & \\ \hline W_4 & \\ \hline W_5 & \\ \hline \end{array} \in \begin{array}{|c|c|} \hline \mathbb{R} & \mathbb{R} \\ \hline [\Lambda_0^{1,1}] & [\Lambda_0^{1,1}] \\ \hline [\Lambda_0^{2,1}] & \\ \hline [\Lambda^{1,0}] & \\ \hline [\Lambda^{1,0}] & \\ \hline \end{array}$$

meaning that the component W_1^+ takes values in \mathbb{R} , and so on. Explicitly, we can write

$$(20) \quad \begin{cases} d\psi^+ = \psi^+ \wedge W_5 + W_2^+ \wedge \omega + W_1^+ \omega^2 \\ d\psi^- = \psi^- \wedge W_5 + W_2^- \wedge \omega + W_1^- \omega^2 \\ d\omega = -\frac{3}{2}W_1^- \psi^+ + \frac{3}{2}W_1^+ \psi^- + W_3 + W_4 \wedge \omega \end{cases}$$

We can do the same for $SU(2)$ -structures on 5-manifolds [7]; the intrinsic torsion now takes values in a 35-dimensional space, and we can arrange its components in the following table:

$$\begin{array}{|c|c|c|} \hline \lambda & & \\ \hline f_1 & f_2 & f_3 \\ \hline g_1^2 & g_1^3 & g_2^3 \\ \hline \beta & & \\ \hline \gamma_1 & \gamma_2 & \gamma_3 \\ \hline \omega^- & & \\ \hline \sigma_1^- & \sigma_2^- & \sigma_3^- \\ \hline \end{array} \in \begin{array}{|c|c|c|} \hline \mathbb{R} & & \\ \hline \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \hline \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \hline \Lambda^1 & & \\ \hline \Lambda^1 & \Lambda^1 & \Lambda^1 \\ \hline \Lambda_-^2 & & \\ \hline \Lambda_-^2 & \Lambda_-^2 & \Lambda_-^2 \\ \hline \end{array}$$

In the above table, Λ^1 is the 4-dimensional representation of $SU(2)$ such that the tangent space at a point is

$$T = \Lambda^1 \oplus \mathbb{R},$$

whereas Λ_-^2 is the 3-dimensional representation of $SU(2)$ consisting of anti-self-dual two-forms on Λ^1 . We shall write, say, $(\omega)_{\Lambda_-^2}$ for the Λ_-^2 component of a two-form ω .

Setting $g_i^j = -g_j^i$, the components of the intrinsic torsion are given by

$$(21) \quad \begin{cases} d\alpha = \alpha \wedge \beta + \sum_{j=1}^3 f^j \omega_j + \omega^- \\ d\omega_i = \gamma_i \wedge \omega_i + \lambda \alpha \wedge \omega_i + \sum_{j \neq i} g_i^j \alpha \wedge \omega_j + \alpha \wedge \sigma_i^- \end{cases}$$

We can now prove the following:

Proposition 5.1. *Define the intrinsic torsion of M as above, and write*

$$(22) \quad W_i = \eta \wedge \Xi_i + \Delta_i, \quad \text{where } \Xi_i = X \lrcorner W_i;$$

then the intrinsic torsion of the quotient is given by

$-\langle \Delta_5, \alpha \rangle$		
$\frac{3}{2}W_1^-$	$-\frac{3}{2}W_1^+ - \frac{1}{2}\langle \Xi_3, \omega_2 \rangle$	$-t^{-1}\Xi_4 - \frac{1}{2}\langle \Xi_3, \omega_3 \rangle$
$-t^{-2}\Xi_5$	$-2t^{-1}W_1^+ - t^{-1}\langle \Xi_2^+, \alpha \rangle$	$-2t^{-1}W_1^- - t^{-1}\langle \Xi_2^-, \alpha \rangle$
$-(\Delta_4)_{\Lambda^1} - \alpha \lrcorner \Xi_3$		
$-(\Delta_5)_{\Lambda^1} - t^{-1}\Xi_2^+ \lrcorner \omega_2$	$-(\Delta_5)_{\Lambda^1} + t^{-1}\Xi_2^- \lrcorner \omega_1$	$(\Delta_4 + d \log t + \frac{1}{2}t\omega_3 \lrcorner \Delta_3)_{\Lambda^1}$
$-(\Xi_3)_{\Lambda^2_-}$		
$-\Delta_2^+$	$-\Delta_2^-$	$t(\alpha \lrcorner \Delta_3 - \phi)_{\Lambda^2_-}$

Proof. Taking the interior product of (20) with X , then substituting (17) in the left-hand side and (18) in the right-hand side, one easily computes:

$$\begin{aligned}
d\alpha &= \frac{3}{2}W_1^-\omega_1 - \frac{3}{2}W_1^+\omega_2 - \Xi_3 - t^{-1}\Xi_4\omega_3 + \Delta_4 \wedge \alpha \\
d\omega_1 &= -\omega_1 \wedge \Delta_5 - t^{-2}\Xi_5\omega_2 \wedge \alpha - t^{-1}\Xi_2^+ \wedge \omega_3 - \Delta_2^+ \wedge \alpha - 2t^{-1}W_1^+\omega_3 \wedge \alpha \\
d\omega_2 &= -\omega_2 \wedge \Delta_5 + t^{-2}\Xi_5\omega_1 \wedge \alpha - t^{-1}\Xi_2^- \wedge \omega_3 - \Delta_2^- \wedge \alpha - 2t^{-1}W_1^-\omega_3 \wedge \alpha
\end{aligned}$$

On the other hand, $d\omega_3 = d \log t \wedge \omega_3 - tX \lrcorner d(\omega \wedge \eta)$ by (17). Using (20) and then (18), we obtain

$$\begin{aligned}
d(\omega \wedge \eta) &= \frac{3}{2}t^{-2}W_1^-\omega_2 \wedge \alpha \wedge \eta + \frac{3}{2}t^{-2}W_1^+\omega_1 \wedge \alpha \wedge \eta + \Delta_3 \wedge \eta + \\
&\quad + t^{-1}\Delta_4 \wedge \omega_3 \wedge \eta + t^{-1}\omega_3 \wedge \phi + \eta \wedge \alpha \wedge \phi ;
\end{aligned}$$

therefore

$$d\omega_3 = d \log t \wedge \omega_3 + \frac{3}{2}t^{-1}W_1^-\omega_2 \wedge \alpha + \frac{3}{2}t^{-1}W_1^+\omega_1 \wedge \alpha + t\Delta_3 + \Delta_4 \wedge \omega_3 - t\alpha \wedge \phi .$$

The decompositions (22) of the components W_3 and W_2^\pm correspond to projections

$$(23) \quad [\Lambda_0^{2,1}] \rightarrow \omega_1^\perp \oplus (\alpha \wedge \omega_1)^\perp , \quad [\Lambda_0^{1,1}] \rightarrow T \oplus \Lambda_-^2 ,$$

respectively; the statement is now a straightforward consequence of (21). \square

Remark. Of the decompositions (23), the first is not surjective; in other words, the components Ξ_3 and Δ_3 are not independent.

It is clear that one can write down a converse to Proposition 5.1, because the quotient determines the intrinsic torsion of M ; one can then characterize the M with special intrinsic torsion in terms of the quotient. For example, in the symplectic half-flat case one obtains this generalization of Lemma 2.2:

Proposition 5.2. *M is symplectic half-flat if and only if the quotient satisfies*

$$\begin{aligned}
(24) \quad d\alpha &= 0 , \quad d\omega_1 = 0 , \quad d\omega_2 \wedge \alpha = t^2\omega_1 \wedge \phi + 2d \log t \wedge \omega_2 \wedge \alpha , \\
d\omega_3 &= d \log t \wedge \omega_3 - t\alpha \wedge \phi .
\end{aligned}$$

Proof. Follows immediately from (18). \square

We now consider the case where M is integrable. In order to state our theorem, we need to introduce two differential operators on 5-manifolds with an

SU(2)-structure. The first one is ∂_α , which maps a function f to $\langle \alpha, df \rangle$. Secondly, consider the endomorphism J_3 of T^*N characterized by

$$J_3\alpha = 0, \quad \omega_1 \wedge \beta = \omega_2 \wedge J_3\beta \text{ for } \beta \in \alpha^\perp;$$

we can then define an operator d^c which maps a function f to $d^c f = J_3 df$.

Theorem 5.3. *If the SU(3)-structure on M is integrable, the function t is a solution of*

$$(25) \quad \partial_\alpha^2 \log t - (\partial_\alpha \log t)^2 - 2t^{-1} \|(d \log t)_{\Lambda^1}\|^2 = 0.$$

The intrinsic torsion is determined by t as follows: α , ω_1 and ω_2 are closed, and ω_3 satisfies

$$(26) \quad d\omega_3 = (d \log t)_{\Lambda^1} \wedge \omega_3 + \frac{1}{\partial_\alpha t} \alpha \wedge (2d \log t \wedge d^c \log t - dd^c \log t)_{\Lambda_-^2};$$

moreover the “curvature form” is

$$(27) \quad \phi = t^{-1} \partial_\alpha \log t \omega_3 - \frac{1}{t^2 \partial_\alpha \log t} (2d \log t \wedge d^c \log t - dd^c \log t)_{\Lambda_-^2} - 2t^{-2} \alpha \wedge d^c \log t.$$

Conversely, let N be a 5-manifold with an SU(2)-structure (α, ω_i) and a function t , where α , ω_1 and ω_2 are closed, and (25), (26) are satisfied. Then the two-form ϕ defined by (27) is closed; if the cohomology class $[\frac{\phi}{2\pi}]$ is an element of $H^2(N, \mathbb{Z})$, there is a circle bundle over N on which an integrable SU(3)-structure is defined by (18), where η is a connection form such that $d\eta = \phi$.

Locally, Theorem 5.3 is a characterization. Indeed, in the second part one can restrict N to a contractible open subset N' , so that

$$0 = [\phi/2\pi] \in H^2(N', \mathbb{Z}).$$

Proof. It is clear from (17) that α , ω_1 and ω_2 are closed. Hence, the only non-vanishing components of the intrinsic torsion are γ_3 and σ_3^- , determined by

$$d\omega_3 = \gamma_3 \wedge \omega_3 + \alpha \wedge \sigma_3^-.$$

Using (18), we find

$$t^{-1} \omega_3 \wedge (\gamma_3 - d \log t) + \alpha \wedge (t^{-1} \sigma_3^- + \phi) = 0.$$

Hence $\gamma_3 = (d \log t)_{\Lambda^1}$, and the component of ϕ in $\Lambda^2(\alpha^\perp)$ is determined by

$$\langle \phi, \omega_1 \rangle = 0 = \langle \phi, \omega_2 \rangle, \quad \langle \phi, \omega_3 \rangle = 2t^{-1} \partial_\alpha \log t, \quad (\phi)_{\Lambda_-^2} = -t^{-1} \sigma_3^-.$$

It also follows from (18) that

$$\begin{aligned} \omega_1 \wedge \phi + 2t^{-3} dt \wedge \omega_2 \wedge \alpha &= 0 \\ \omega_2 \wedge \phi - 2t^{-3} dt \wedge \omega_1 \wedge \alpha &= 0 \end{aligned}$$

Therefore

$$\alpha \lrcorner \phi = -2t^{-2} d^c \log t.$$

For brevity, we set $s = \partial_\alpha \log t$. By construction ϕ is closed, so

$$(28) \quad 0 = t^{-1} (-s d \log t \wedge \omega_3 + ds \wedge \omega_3 + s(d \log t - s \alpha) \wedge \omega_3 + s \alpha \wedge \sigma_3^- + d \log t \wedge \sigma_3^- - d \sigma_3^-) + 2t^{-2} \alpha \wedge (-2d \log t \wedge d^c \log t + dd^c \log t).$$

We can split (28) into two equations by taking the wedge and the interior product with α . One of these is satisfied automatically: indeed, taking d of $d\omega_3$ we find

$$0 = \alpha \wedge (d \log t \wedge \sigma_3^- - d\sigma_3^- + ds \wedge \omega_3) ,$$

so the right-hand side of (28) vanishes on wedging with α . Taking the interior product gives

$$(\partial_\alpha s - s^2)\omega_3 + 2s\sigma_3^- + 2t^{-1}(-2d \log t \wedge d^c \log t + dd^c \log t)_{\Lambda^2(\alpha^\perp)} .$$

It is now clear that σ_3^- can be expressed in terms of t , giving (26). Using the general formula

$$\langle \beta \wedge J_3 \beta, \omega_3 \rangle = \|\beta\|^2 - \langle \beta, \alpha \rangle^2 ,$$

we also deduce that t satisfies (25).

Conversely, suppose (26) and (25) are satisfied, and define ϕ by (27). The above calculations show that ϕ is closed and the construction of Proposition 4.2 defines an integrable $SU(3)$ -structure. \square

Remark. The condition of Theorem 5.3 implies in particular that 28 out of the 35 components of the intrinsic torsion of (N, α, ω_i) vanish. A similar construction was described in [1], starting with a 7-manifold with holonomy G_2 , and defining an $SU(3)$ -structure on the quotient. In that case, the vanishing components of the intrinsic torsion of the quotient are also 28, though out of 42.

In general (25) and (26) are not independent, because the norm on one-forms depends on ω_3 . Motivated by this observation, we consider the special case

$$(d \log t)_{\Lambda^1} = 0 ;$$

in order to apply Theorem 5.3, we have to assume that the $SU(2)$ -structure is integrable. Let x be a coordinate in the direction of α , so that $\alpha = dx$. Set $t = (1 - x)^{-1}$; then (25) is satisfied. Suppose that one has a circle bundle over N with Chern class $[\frac{1}{2\pi}\omega_3]$; then the hypotheses of Theorem 5.3 hold. Define a connection form η such that $d\eta = \omega_3$; then

$$\omega = (1 - x)\omega_3 + \eta \wedge \alpha , \quad \Psi = (\omega_1 + i\omega_2) \wedge (\eta + i(1 - x)^2 \alpha) ,$$

defines an integrable $SU(3)$ -structure. One can actually prove that if the original 5-manifold has holonomy $SU(2)$, then the Calabi-Yau 6-manifold has holonomy $SU(3)$.

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